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VI. *Addition to the Memoir on TSCHIRNHAUSEN'S Transformation.*

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In the memoir “On TSCHIRNHAUSEN'S Transformation,” Philosophical Transactions, vol. clii. (1862) pp. 561–568, I considered the case of a quartic equation: viz. it was shown that the equation

$$(a, b, c, d, e)(x, 1)^4 = 0$$

is, by the substitution

$$y = (ax + b)B + (ax^2 + 4bx + 3c)C + (ax^3 + 4bx^2 + 6cx + 3d)D,$$

transformed into

$$(1, 0, \mathbb{C}, \mathbb{D}, \mathbb{E})(y, 1)^4 = 0,$$

where $(\mathbb{C}, \mathbb{D}, \mathbb{E})$ have certain given values. It was further remarked that $(\mathbb{C}, \mathbb{D}, \mathbb{E})$ were expressible in terms of U' , H' , Φ' , invariants of the two forms $(a, b, c, d, e)(X, Y)^4$, $(B, C, D)(Y, -X)^2$, of I, J , the invariants of the first, and of $\Theta' = BD - C^2$, the invariant of the second of these two forms,—viz. that we have

$$\mathbb{C} = 6H' - 2I\Theta',$$

$$\mathbb{D} = 4\Phi',$$

$$\mathbb{E} = IU'^3 - 3H'^2 + I^2\Theta'^2 + 12J'\Theta'U' + 2I'\Theta'H'.$$

And by means of these I obtained an expression for the quadrinvariant of the form

$$(1, 0, \mathbb{C}, \mathbb{D}, \mathbb{E})(y, 1)^4;$$

viz. this was found to be

$$= IU'^2 + \frac{4}{3}I^2\Theta'^2 + 12J\Theta'U'.$$

But I did not obtain an expression for the cubinvariant of the same function: such expression, it was remarked, would contain the square of the invariant Φ' ; it was probable that there existed an identical equation,

$$JU'^3 - IU'^2H' + 4H'^3 + M\Theta' = -\Phi'^2,$$

which would serve to express Φ'^2 in terms of the other invariants; but, assuming that such an equation existed, the form of the factor M remained to be ascertained; and until this was done, the expression for the cubinvariant could not be obtained in its most simple form. I have recently verified the existence of the identical equation just referred to, and have obtained the expression for the factor M ; and with the assistance of this identical equation I have obtained the expression for the cubinvariant of the form

$$(1, 0, \mathbb{C}, \mathbb{D}, \mathbb{E})(y, 1)^4.$$

The expression for the quadrinvariant was, as already mentioned, given in the former memoir: I find that the two invariants are in fact the invariants of a certain linear function of U, H ; viz. the linear function is $=U'U + \frac{2}{3}\Theta'H$; so that, denoting by I^* , J^* , the quadrinvariant and the cubinvariant respectively of the form

$$(1, 0, \mathbb{C}, \mathbb{D}, \mathbb{E} \mathcal{Y} y, 1)^4,$$

we have

$$I^* = \tilde{I}(U'U + 4\Theta'H),$$

$$J^* = \tilde{J}(U'U + 4\Theta'H),$$

where \tilde{I} , \tilde{J} signify the functional operations of forming the two invariants respectively. The function $(1, 0, \mathbb{C}, \mathbb{D}, \mathbb{E} \mathcal{Y} y, 1)^4$, obtained by the application of TSCHIRNHAUSEN'S transformation to the equation

$$(a, b, c, d, e \mathcal{Y} x, 1)^4 = 0,$$

has thus the *same invariants* with the function

$$U'U + 4\Theta'H = U'(a, b, c, d, e \mathcal{Y} x, 1)^4 + 4\Theta'(ac - b^2, ad - bc, ae + 2bd - 3c^2, be - cd, ce - d^2 \mathcal{Y} x, 1)^4,$$

and it is consequently a linear transformation of the last-mentioned function; so that the application of TSCHIRNHAUSEN'S transformation to the equation $U=0$ gives an equation linearly transformable into, and thus virtually equivalent to, the equation

$$U'U + 4\Theta'H = 0,$$

which is an equation involving the single parameter $\frac{4\Theta'}{U'}$: this appears to me a result of considerable interest. It is to be remarked that TSCHIRNHAUSEN'S transformation, wherein y is put equal to a rational and integral function of the order $n-1$ (if n be the order of the equation in x), is not really less general than the transformation wherein y is put equal to any rational function $\frac{V}{W}$ whatever of x ; such rational function may, in fact, by means of the given equation in x , be reduced to a rational and integral function of the order $n-1$; hence in the present case, taking V, W to be respectively of the order $n-1$, $=3$, it follows that the equation in y obtained by the elimination of x from the equations

$$(a, b, c, d, e \mathcal{Y} x, 1)^4 = 0,$$

$$y = \frac{(\alpha, \beta, \gamma, \delta \mathcal{Y} x, 1)^3}{(\alpha', \beta', \gamma', \delta' \mathcal{Y} x, 1)^3}$$

is a mere linear transformation of the equation $AU + BH = 0$, where A, B are functions (not as yet calculated) of $(a, b, c, d, e, \alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta')$.

Article Nos. 1, 2, 3.—*Investigation of the identical equation*

$$JU'^3 - IU'^2H' + 4H'^3 + M\Theta' = -\Phi'^2.$$

1. It is only necessary to show that we have such an equation, M being an invariant,

in the particular case $a=e=1$, $b=d=0$, $c=\theta$, that is for the quartic function $(1, 0, \theta, 0, 1)(x, 1)^4$; for, this being so, the equation will be true in general. Writing the equation in the form

$$-M\Theta' = U'^2(JU' - IH') + 4H'^3 + \Phi'^2,$$

and observing that we have

$$\begin{aligned} U' &= (B^2 + D^2) + 2\theta BD + 4\theta C^2, \\ H' &= \theta(B^2 + D^2) + (1 + \theta^2)BD - 4\theta^2 C^2, \\ \Theta' &= BD - C^2, \\ \Phi' &= (1 - 9\theta^2)C(B^2 - D^2), \\ I &= 1 + 3\theta^2, \\ J &= \theta - \theta^3, \end{aligned}$$

and thence

$$JU' - IH' = -4\theta^3(B^2 + D^2) + (-1 - 2\theta^2 - 5\theta^4)BD + (8\theta^2 + 8\theta^4)C^2,$$

the equation becomes

$$\begin{aligned} -(BD - C^2)M &= \\ &\quad \{ -4\theta^3(B^2 + D^2) + (-1 - 2\theta^2 - 5\theta^4)BD + (8\theta^2 + 8\theta^4)C^2 \} \\ &\quad \times \{ B^2 + D^2 + 2\theta BD + 4\theta C^2 \}^2 \\ &\quad + 4\{ \theta(B^2 + D^2) + (1 + \theta^2)BD - 4\theta^2 C^2 \}^3 \\ &\quad + (1 - 9\theta^2)^2 C^2 \{ (B^2 + D^2)^2 - 4B^2 D^2 \}. \end{aligned}$$

2. It is found by developing that the right-hand side is in fact divisible by $BD - C^2$, and that the quotient is

$$\begin{aligned} &= (-1 + 10\theta^2 - 9\theta^4)(B^2 + D^2)^2 \\ &\quad + (8\theta + 16\theta^3 - 24\theta^5)(B^2 + D^2)BD \\ &\quad + (4 + 8\theta^2 + 4\theta^4 - 16\theta^6)B^2 D^2 \\ &\quad + (-64\theta^3 - 192\theta^5)(B^2 + D^2)C^2 \\ &\quad + (16\theta^2 - 416\theta^4 - 112\theta^6)BDC^2 \\ &\quad + (-128\theta^4 + 128\theta^6)C^4. \end{aligned}$$

3. This is found to be

$$\begin{aligned} &= -I^2 U'^2 + 12JU'H' + 4IH'^2 \\ &\quad - 8IJU'\Theta' \\ &\quad - 16J^2\Theta'^2, \end{aligned}$$

which is consequently the value of $-M$. We have therefore

$$\begin{aligned} -\Phi'^2 &= JU'^3 - IU'^2H' + 4H'^3 \\ &\quad + (I^2 U'^2 - 12JU'H' - 4IH'^2)\Theta' \\ &\quad + 8IJU'\Theta'^2 \\ &\quad + 16J^2\Theta'^3, \end{aligned}$$

which is the required identical equation.

Article No. 4.—*Calculation of the Cubinvariant.*

4. We have

$$\begin{aligned}
 J^* &= \frac{1}{6} \mathbb{C} \cdot \mathbb{E} - \left(\frac{1}{6} \mathbb{C}\right)^3 - \left(\frac{1}{4} \mathbb{D}\right)^2 \\
 &= (H - \frac{1}{3} I \Theta') \{ I U'^2 - 3 H'^2 + (12 J U' + 2 I H') \Theta' + I^2 \Theta'^2 \} \\
 &\quad - (H - \frac{1}{3} I \Theta')^3 \\
 &\quad - \Phi'^2,
 \end{aligned}$$

whence, substituting for $-\Phi'^2$ its value and reducing, we find

$$J^* = J U'^3 + \Theta' \cdot \frac{2}{3} I^2 U'^2 + \Theta'^2 (4 I J U') + \Theta'^3 (16 J^2 - \frac{8}{27} I^3).$$

Article No. 5.—*Final expressions of the two Invariants.*

The value of I^* has been already mentioned to be $I^* = I U'^2 + \Theta' 12 J U' + \Theta'^2 \cdot \frac{4}{3} I^2$, and it hence appears that the values of the two invariants may be written

$$\begin{aligned}
 I^* &= (I, 18 J, 3 I^2 \mathbb{X} U', \frac{2}{3} \Theta')^2, \\
 J^* &= (J, I^2, 9 I J, -I^3 + 54 J^2 \mathbb{X} U', \frac{2}{3} \Theta')^3.
 \end{aligned}$$

But we have (see Table No. 72 in my "Seventh Memoir on Quantics" †)

$$\begin{aligned}
 \tilde{I}(\alpha U + 6 \beta H) &= (I, 18 J, 3 I \mathbb{X} \alpha, \beta)^2 \\
 \tilde{J}(\alpha U + 6 \beta H) &= (J, I^2, 9 I J, -I^3 + 54 J^2 \mathbb{X} \alpha, \beta)^3;
 \end{aligned}$$

so that, writing $\alpha = U'$, $\beta = \frac{2}{3} \Theta'$, we have

$$\begin{aligned}
 I^* &= \tilde{I}(U' U + 4 \Theta' H), \\
 J^* &= \tilde{J}(U' U + 4 \Theta' H);
 \end{aligned}$$

or the function $(1, 0, \mathbb{C}, \mathbb{D}, \mathbb{E} \mathbb{X} y, 1)^4$ obtained from TSCHIRNHAUSEN's transformation of the equation $U=0$ has the same invariants with the function $U' U + 4 \Theta' H$; or, what is the same thing, the equation $(1, 0, \mathbb{C}, \mathbb{D}, \mathbb{E} \mathbb{X} y, 1^4)=0$ is a mere linear transformation of the equation $U' U + 4 \Theta' H = 0$; which is the above-mentioned theorem.

† Philosophical Transactions, vol. cl. (1861), pp. 277–292.